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FOUR METHODS OF SOLVING FOR THE SPHERICAL ERROR PROBABLE ASSOCIATED WITH A THREE-DIMENSIONAL NORMAL DISTRIBUTION

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Holloman AFB, New Mexico

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TECHNICAL REPORT

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Prepared by

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FOREWORD

This document has been prepared in support of a study conducted by the Directorate of Foreign Technology. The study was documented under Program Element 6540221F, System Number 873A, Project N105, entitled "Foreign Technology for Guidance." The programing and computer time were supplied by the Directorate of Technical Support and charged to Program Element 6540221F, System Number 873A, Mission Workload Number 87300, entitled "Foreign Technology Analysis (Computer Support)."

PUBLICATION REVIEW

This technical report has been reviewed and is approved.

FOR THE COMMANDER

FREDERICK E. BUSH	WILLIAM C. McCORMICK
Lt Col, USAF	Lt Col, USAF
Director of Foreign Technology	Director of Technical Support

ABSTRACT

When the predicted position of a satellite contains normally distributed errors, the position uncertainty can be described by a Spherical Error Probable or SEP. The SEP is calculated by integrating the three-dimensional normal probability density function over a spherical volume. The SEP is set equal to the radius of that volume which contains the satellite with 50% probability. In this report the authors present four methods for integrating the density function and finding the SEP. The three normal variates in the density function are assumed to be independent and unbiased with known variances.

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LIST OF ABBREVIATIONS AND SYMBOLS

x, y, z.	Denote the three axes of an orthogonal coordinate system. Also denote three random errors in position measured along the orthogonal axes.
$\sigma x, \sigma y, \sigma z$	Denote the standard deviations of the position errors measured along the x, y and z axes, respectively.
R	Radius of a spherical volume.
SEP	Spherical Error Probable.
R_o	The first estimate for the SEP.
R_1	The second estimate for the SEP.
R_i	The $i + 1^{st}$ estimate for the SEP.
ΔR_i	An incremental change in the radius R_i.
P	Probability.
P_i	The probability that a point lies in a sphere of radius R_i.
ΔP_i	An incremental change in the probability P_i.
$\partial P/\partial R$	The partial derivative of P with respect to R.
$(\partial P/\partial R)_i$	The partial derivative of P with respect to R evaluated at R_i.
L, K, H	The predicted position coordinates of a point in space.
CEP	Circular Error Probable.
q	The standard normal variate.
ρ	The radial axis in a spherical coordinate system.

$\alpha_1(C_2, n)$	A number calculated from the recursion formulas in Appendix B.
j, n	Summation indices.

SECTION 1

INTRODUCTION

1.1 Purpose of the Report

In this paper we report four methods of solving Eq. (1-1) for R when σ_x, σ_y and σ_z are known and P is set equal to 0.5.

$$P = \frac{1}{(2\pi)^{3/2}\sigma_x \sigma_y \sigma_z} \int_{-R}^{R} \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \int_{-\sqrt{R^2-z^2-y^2}}^{\sqrt{R^2-z^2-y^2}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right)\right] dx\, dy\, dz \qquad (1\text{-}1)$$

Eq. (1-1) describes the integration of an unbiased, three-dimensional, normal probability density function over a spherical volume. R is the radius of the sphere; x, y and z are independent random variables, and P is a probability. For P set equal to 0.5, we define R as the Spherical Error Probable (SEP).

1.2 Uses for the SEP

The SEP is useful as a measure of position uncertainty in three dimensions. It is a single number, like 3250 ft, describing the uncertainty in the location of one vehicle at a given time. The SEP is applicable to problems in satellite tracking, missile defense, submarine navigation and air traffic control.[5][6][7] As an example,

we describe in the next four paragraphs a SEP that might be computed in predicting the location of a manned orbiting laboratory.

A manned orbiting laboratory in a near-earth orbit requires resupply. The supply mission is to be flown by a second missile, the transporter, and a rendezvous and docking maneuver is to be executed when the transporter reaches the laboratory. A successful mission hinges on an accurate prediction of the laboratory's location at the time of rendezvous. The mission planners are asked to predict the laboratory's future position by using radar measurements made just before the transporter is launched.

The mission planners would like to make an absolute prediction like the following:

> "The laboratory will be at latitude 20° N, longitude 35° W and 100 nm altitude at 2117 Greenwich Mean Time."

They cannot make such a statement, however, because the radar data contains normally distributed errors. The present location of the laboratory is uncertain so its future position cannot be predicted exactly; the predicted latitude, longitude and altitude contain the unknown random errors x, y and z, respectively.

The mission planners choose to compute a SEP to describe the uncertainty in the future position of the laboratory. Their computed SEP is 8000 feet. The mission planners use the SEP to make the following

probablistic statement to the transporter crew:

> "The laboratory will be within 8000 feet of 20° N latitude, 35° W longitude and 100 nm altitude at 2117 Greenwich Mean Time with 50% probability."

The transporter crew thus has an estimate for the uncertainty in the laboratory's location. Upon reaching the predicted point of rendezvous, the crew will probably have to search through a sphere of at least 8000 foot radius to find the laboratory.

The previous example illustrates the information implied in the Spherical Error Probable. If a vehicle is said to be located at point L, K and H with a SEP of 8000 feet, we immediately know that:

(1) The location of the vehicle is uncertain.

(2) The three position coordinates, L, K and H, contain normally distributed errors.

(3) The probability density function is so shaped that 50% of the probability is contained in a sphere of 8000 foot radius. The sphere is centered on the mean predicted location of the vehicle.

A SEP can be formulated to describe position uncertainties for many vehicles - satellites, aircraft, submarines, and missiles. In this report, however, we restrict the SEP application to those three-dimensional problems where the random position errors are normally distributed, unbiased and independent.

1.3 The Three-Dimensional Normal Distribution

If three random variables, x, y and z, are independent, unbiased and normally distributed, their joint three-dimensional probability density function is given by the equation[1]:

$$f(x, y, z) = \frac{1}{(2\pi)^{3/2}\sigma_x \sigma_y \sigma_z} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right)\right] \quad (1\text{-}2)$$

If the density function, f (x, y, z), is integrated over a closed volume, a probability, P, results. If the three random variables describe the position of a point, P is the probability that the point is located somewhere in the volume. In this report the closed volume is a sphere. R is defined as the radius of the sphere and Eq. (1-1) describes the probability integral for P[2]:

$$P = \frac{1}{(2\pi)^{3/2}\sigma_x \sigma_y \sigma_z} \int_{-R}^{R} \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \int_{-\sqrt{R^2-z^2-y^2}}^{\sqrt{R^2-z^2-y^2}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right)\right] dx\, dy\, dz$$

(1-1)

When the standard deviations, σ_x, σ_y, σ_z, are known and R is specified, Eq. (1-1) can be solved for P.[1][2] We, however, are interested in the complementary problem. We seek a solution for R when P is given. In particular we want R when σ_x, σ_y and σ_z are

known and P is set equal to 0.5. This R is defined as the Spherical Error Probable or the SEP.

The probability limit, in this case 0.5 or 50%, is not unique. Any limit, say 95%, can be used in the definition of the SEP. The 50% limit, however, is consistent with the definition of the two-dimensional Circular Error Probable (CEP) used in missile accuracy studies.[3]

1.4 Outline of the Paper

In the next four sections we outline four methods for computing the SEP when σ_x, σ_y and σ_z are given. In Section 2, for example, we use the CEP curve to approximate the SEP in the special case when $\sigma_x = \sigma_y$. In Section 3 a paper and pencil solution is developed. Section 4 describes a computer method for finding the SEP, and Section 5 summarizes the computer data in a graphical solution.

The paper closes with three appendices. The first appendix extends the approximation method of Section 2 to the cases where $\sigma_x \neq \sigma_y$. The second appendix outlines H. W. Lilliefors' solution of Eq. (1-1). The third appendix contains a computer program and sample data.

The reader with standard deviations, σ_x, σ_y and σ_z, in hand and a deadline to meet should proceed directly to Fig. 5.2 in Section 5.

For a wide range of standard deviations, Fig. 5.2 permits a direct reading of the SEP.

SECTION 2

THE FIRST SOLUTION: A METHOD OF ESTIMATING THE SEP

2.1 Outline of the Estimation Method

In general, the SEP cannot be found by a direct integration of Eq. (1-1). We cannot write a mathematical equation that expresses R in terms of P, σ_x, σ_y and σ_z. We can, however, estimate the SEP by looking at special cases. In this section we show how the SEP can be estimated when σ_x and σ_y are equal and

(1) Small compared to σ_z.

(2) Large compared to σ_z.

(3) Equal to σ_z.

(4) Equal to one-half of σ_z.

Here we also introduce the normalized function, SEP/σ_z, because it is more easily plotted than the SEP.

2.2 The SEP When σ_x and σ_y are Equal and Small

In Sec. 1.3 we wrote down Eq. (1-2), the three-dimensional normal density function:

$$f(x, y, z) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right)\right] \qquad (1\text{-}2)$$

If two of the standard deviations in this expression are equal and smaller than the third deviation, the three-dimensional distribution

look$ one-dimensional. For example, if σ_x and σ_y are equal and smaller than σ_z, the three-dimensional distribution looks like a one-dimensional distribution in the variate z. In the limit, as the ratio σ_x/σ_z approaches zero, the SEP calculation reduces to the problem of finding the half-length of a straight line. The full-length of the straight line includes the location of the random point with 50% probability.

The probability equation to be solved is just:

$$P = \frac{1}{\sqrt{2\pi}\,\sigma_z} \int_{-R}^{R} \exp\left[-\frac{z^2}{2\sigma_z^2}\right] dz \qquad (2-1)$$

We seek R when σ_z is given and P is set equal to 0.5.

If q is substituted for z/σ_z, the integrand in Eq. (2-1) becomes the standard normal density function for which there are tabulated solutions.[4] Writing Eq. (2-1) in the standard form we get:

$$P = \frac{1}{\sqrt{2\pi}} \int_{-R/\sigma_z}^{R/\sigma_z} \exp\left[-\frac{1}{2}q^2\right] dq \qquad (2-2)$$

We use the tables and find that when $R/\sigma_z = 0.674$, P equals 0.5. Thus, for σ_x and σ_y equal to zero, the ratio SEP/σ_z has the value 0.674. This result is plotted in Figure 2.1, Sec. 2.6, as a point

on the ordinate where $\sigma_x/\sigma_z = 0$. This point is plotted as the first step in constructing a curve of SEP/σ_z versus σ_x/σ_z under the special condition that σ_x equals σ_y.

2.3 The SEP When σ_x and σ_y are Equal and Large

We have seen that the three-dimensional density function looks one-dimensional when two of the standard deviations are equal and small compared to the third. If, conversely, two of the standard deviations are equal and large compared to the third deviation, the distribution looks two-dimensional. If σ_x and σ_y are equal and much larger than σ_z, the density function looks two-dimensional in the variates x and y.

Calculation of the SEP reduces to the problem of finding the radius of a circle. The circular area contains the location of the random point with 50% probability. The desired radius is commonly called the Circular Error Probable (CEP). The CEP is usually used to specify the probable impact error of a long range missile.

The two-dimensional probability integral is given by Eq. (2-3).

$$P = \frac{1}{2\pi\,\sigma_x\,\sigma_y} \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right] dx\, dy \qquad (2\text{-}3)$$

A solution for this integral is graphed in Ref. 3. For σ_y/σ_x greater than 1/3 the SEP is approximated by Eq. (2-4).

$$SEP = 0.59 (\sigma_x + \sigma_y) \tag{2-4}$$

In our case, with σ_y and σ_x equal and much larger than σ_z, the SEP is given by the expression

$$SEP = 1.18 \sigma_x = 1.18 \sigma_y \tag{2-5}$$

Equation (2-5) is made more convenient if we divide both sides of the equation by σ_z.

$$\frac{SEP}{\sigma_z} = 1.18 \frac{\sigma_x}{\sigma_z} \tag{2-6}$$

This result is plotted in Fig. 2.1 as a straight line with slope equal to 1.18 and intercept at $\sigma_x/\sigma_z = 0$. The curve for $\sigma_x = \sigma_y$, in Fig. 5.2, Sec. 5, can be used to verify that this expression for the SEP is accurate when the ratio σ_x/σ_z is greater than approximately 4.5.

2.4 The SEP when σ_x, σ_y and σ_z are All Equal

When all three standard deviations, σ_x, σ_y and σ_z, are equal, the probability density function retains its three-dimensional form. We, however, can simplify the integral equation for P by changing to spherical coordinates and substituting ρ^2 for $x^2 + y^2 + z^2$ in the integrand. The new expression is then:

$$P = \frac{1}{(2\pi)^{3/2}\sigma_z^3} \int_0^R \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \exp\left[-\frac{\rho^2}{2\sigma_z^2}\right] \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \qquad (2\text{-}7)$$

J. S. Toma, in Ref. 1, reports that if P = 0.5, Eq. (2-7) can be solved to give the result that the

$$SEP = 1.5382\sigma_z \qquad (2\text{-}8)$$

If we divide both sides of this expression by σ_z, Eq. (2-9) appears:

$$\frac{SEP}{\sigma_z} = 1.5382 \qquad (2\text{-}9)$$

Thus, for $\sigma_x = \sigma_y = \sigma_z$ we have the result that SEP/ σ_z = 1.5382. This result is plotted in Fig. 2.1 as a point at $\sigma_x/\sigma_z = 1$.

2.5 The SEP when $\sigma_x = \sigma_y = 0.5\,\sigma_z$

The last special case we examine is that for σ_x and σ_y equal to $0.5\sigma_z$. The SEP is calculated using the work of H. W. Lilliefors in Ref. 2. He plots the probability, P, versus σ_y when R = 1. His Fig. 2 shows that P = 0.5 when $\sigma_x = 0.5$, $\sigma_y = 0.5$ and $\sigma_z = 1$. Thus, for $\sigma_x/\sigma_z = 0.5$, the ratio, SEP/σ_z, must be equal to one.

2.6 The Predicted SEP Curve for σ_x and σ_y Equal

In Sec. 2.2 through 2.5 we have computed one line and three discrete points which describe the behavior of the function SEP/σ_z. We

have plotted these points and the line in Fig. 2.1 and we now draw in a dotted line to complete the curve. This graph can be used to estimate the SEP when σ_x and σ_y are equal.* The reader should note this

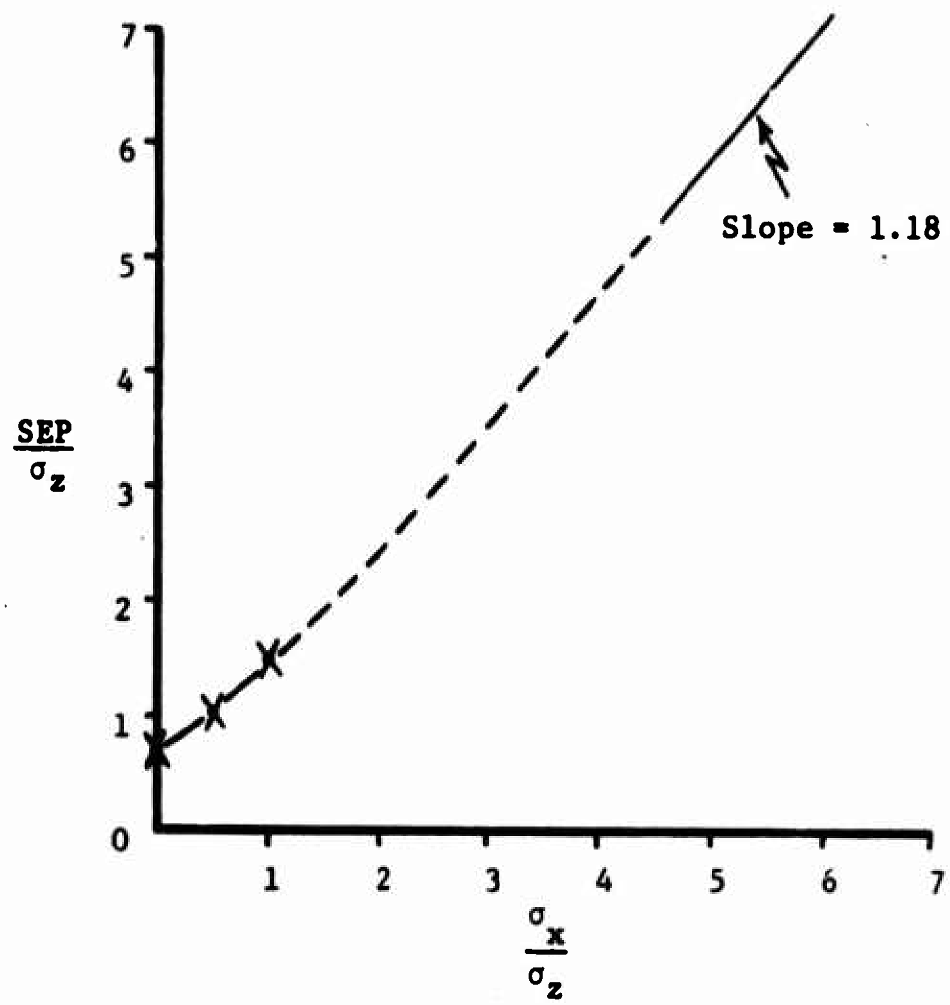

Figure 2.1 The predicted SEP/σ_z curve when $\sigma_x = \sigma_y$.

special condition. When σ_x and σ_y are not equal, the SEP must be estimated by the methods of Appendix A.

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* The curve can be used to compute the SEP when any two standard deviations are equal. The reader need only interchange the subscripts, x, y and z, on the standard deviations.

To find a SEP given $\sigma_x = \sigma_y$, σ_z and Fig. 2. 1, we first find the ratio σ_x/σ_z. If the ratio has the value 2.25, for example, we enter Fig. 2. 1 at the bottom where $\sigma_x/\sigma_z = 2.25$. We proceed vertically upward to the curve and read that the SEP/σ_z has the value 2.85. The SEP is then computed by multiplying 2.85 by the standard deviation, σ_z.

A more exact solution can be found by using paper, pencil, a desk calculator and the iterative method of Sec. 3.

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SECTION 3

THE SECOND SOLUTION: AN ITERATIVE METHOD OF FINDING THE SEP

3.1 Outline of the Method

Given one set of standard deviations and P set equal to 0.5, a SEP can be calculated using paper, pencil and a desk calculator. The method uses H. W. Lilliefors' solution for Eq. (1-1) and we solve by iteration for the SEP.

3.2 Lilliefors' Solution

Lilliefors' solution (see Appendix B) for the three dimensional probability, P, has the form:[1]

$$P = \sqrt{2/\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} \alpha_1(C_2, n)}{2^{(n+1)} \sigma_y^{(2n-1)} \sigma_x \sigma_z} \cdot \sum_{j=0}^{\infty} \left\{ \left[\frac{(-1)^j}{j!(2\sigma_z^2)^j} \right] \left[\frac{1}{(j+n+\frac{1}{2})(j+n-\frac{1}{2})\ldots(j+\frac{1}{2})} \right] \right\}. \qquad (3\text{-}1)$$

This solution is valid when R = 1. For R not equal to one, Lilliefors' solution has the revised form:

$$P = \sqrt{2/\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} \alpha_1(C_2, n)}{2^{(n+1)} \left(\frac{\sigma_y}{R}\right)^{(2n-1)} \left(\frac{\sigma_x}{R}\right)\left(\frac{\sigma_z}{R}\right)} \cdot \sum_{j=0}^{\infty} \left\{ \left[\frac{(-1)^j}{j!\left(2\frac{\sigma_z^2}{R^2}\right)^j} \right] \left[\frac{1}{(j+n+\frac{1}{2})(j+n-\frac{1}{2})\ldots(j+\frac{1}{2})} \right] \right\} \qquad (3\text{-}2)$$

To find the SEP, we set P equal to 0.5 and solve for R. The equation, however, is too complicated to be solved in closed form. We use an iterative procedure.

3.3 The Iterative Procedure

The first step in the iteration is to specify σ_x, σ_y and σ_z. Next, we make an initial estimate for the SEP, say $R = R_o$. R_o, σ_x, σ_y and σ_z are then substituted into Eq. (3-2) and we solve for P. If each of the normalized standard deviations, σ_x/R_o, σ_y/R_o and σ_z/R_o, is greater than about 0.4, the two infinite series in Eq. (3-2) can each be truncated after 10 terms.

The solution proceeds by comparing the computed P, say it's P_o, with 0.5. If P_o is equal to 0.5, we define the initial estimate, R_o, as the SEP. The solution is deemed complete for the given values of σ_x, σ_y and σ_z. If, however, P_o is not equal to 0.5, we make a new estimate for the SEP, say $R = R_1$, and repeat the procedure.

If the reader has one set of standard deviations, paper, pencil and some patience, this procedure will produce a SEP in a few hours. The calculation time is reduced if you have a good first estimate for the SEP. If, however, many sets of standard deviations are to be used, the time required to calculate the exact SEPs is enormous. In this case we recommend the computer solution in Section 4.

SECTION 4

THE THIRD SOLUTION: A COMPUTER METHOD OF FINDING THE SEP

4.1 Outline of the Method

The computer solution for the SEP mechanizes the iterative procedure outlined in Sec. 3.* The computer method, however, forces us to deal with three mathematical problems that we ignored in Sec. 3. First, we have to identify the sets of standard deviations for which Lilliefors' series solution will not converge. Second, we must find alternate methods for finding the SEP when the series solution diverges. Third, we need to write a mathematical rule for estimating the SEP.

4.2 Conditions for Convergence

H. W. Lilliefors' series solution for P is used in the search for the SEP. The revised form of Lilliefors' solution is given by Eq. (4-1).

$$P = \sqrt{2/\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}\, \alpha_1(C_2, n)}{2^{(n+1)} \left(\frac{\sigma_y}{R}\right)^{(2n-1)} \left(\frac{\sigma_x}{R}\right)\left(\frac{\sigma_z}{R}\right)} \cdot \sum_{j=o}^{\infty} \left\{ \left[\frac{(-1)^j}{j!\left(2\,\frac{\sigma_z^2}{R^2}\right)^j} \right] \left[\frac{1}{(j+n+\frac{1}{2})\,(j+n-\frac{1}{2})\ldots(j+\frac{1}{2})} \right] \right\} \tag{4-1}$$

---

* A listing of the computer solution is presented in Appendix C.

When any one of the normalized standard deviations, σ_x/R, σ_y/R or σ_x/R, is less than approximately 0.2, Eq. (4-1) will not converge to a meaningful value for P.* If, for example, $\sigma_z/SEP = 0.11$ and 100 terms are used in each series, Eq. (4-1) may give a probability like 2×10^6, or -0.93, or 25.7. None of these numbers are admissible values for P. The probability can only have a value between zero and one. When calculating the SEPs for many sets of standard deviations, we must identify those sets of standard deviations for which the computer solution will not converge.

Inadmissible sets of standard deviations are identified by testing the ratios σ_x/SEP, σ_y/SEP and σ_z/SEP against 0.2, the convergence limit. We form the ratios by dividing the standard deviations by an estimate for the SEP. The SEP is estimated by the methods of Sec. 2 or Appendix A. If one of the ratios, σ_z/SEP for example, is less than 0.2, we conclude that Eq. (4-1) cannot be used to find the exact SEP for that set of standard deviations. We look for another method of finding the SEP for that set.

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* We are not concerned here with absolute convergence or divergence. We define Lilliefors' solution to be convergent if we get a meaningful value for P by using a reasonable number of terms in each of the two infinite series. In this paper we call 100 terms reasonable.

4.3 Alternate Solutions for the SEP

If a set of three deviations contains one element, σ_z for example, that fails the convergence test, we can choose to ignore σ_z and solve for a SEP in two dimensions. The SEP, so calculated, is obviously an approximation to the true SEP since we assume that $\sigma_z = 0$ when, in fact, it is a non-zero positive number. The approximation is good, however, when σ_z/SEP is less than about 0.2.

The two-dimensional SEP can be calculated directly from Eq. (4-2) when the ratio, σ_y / σ_x, is greater than about one-third.[3]

$$\text{SEP} = 0.59\,(\sigma_x + \sigma_y) \qquad (4\text{-}2)$$

The SEP can also be calculated by iteration if we substitute Eq. (4-3) for Eq. (4-1) in the computer solution.[2]

$$P = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}\ \alpha_{1}(C_2,\ n)}{2^n\, n!\, (\sigma_y/R)^{(2n-1)}(\sigma_x/R)} \qquad (4\text{-}3)$$

The computer program in Appendix C incorporates both equations (4-1) and (4-3). The program also contains a rule for selecting R_o, the initial estimate for the SEP. This rule and a method for upgrading the estimate are discussed in the following paragraphs.

4.4 A Rule for Selecting R_o

As noted in Sec. 4.1, the computer program mechanizes an iterative search for the SEP. The search is started, for a given set of

standard deviations, by estimating the SEP. If the first estimate, R_o, causes the probability, P, to equal 0.5, the iterative search is stopped and the SEP is equated to R_o. In the usual case, however, the first estimate for the SEP is wrong and successive estimates must be made to drive P toward 0.5.

The first estimate for the SEP is made by setting R_o equal to the <u>minimum</u> value of the three ratios, $\sigma_x/0.2$, $\sigma_y/0.2$ and $\sigma_z/0.2$. This estimate guarantees that none of the normalized standard deviations, σ_x/R_o, σ_y/R_o and σ_z/R_o, will be less than 0.2 during the first iteration. This choice of R_o avoids the convergence problems discussed in Sec. 4.2.

If R_o, the first estimate for the SEP, is correct, the computer will make one iteration, stop, and print out R_o as the SEP. If the true SEP, however, is much greater than R_o, the computer method will not converge to a solution for the SEP. The computer will make two iterations, test for a diverging solution and stop. When the true SEP is less than R_o, the computer will proceed to a solution for the SEP by making successive estimates for R.

If the first estimate for the SEP is not correct, the second estimate, R_1, is made by adding ΔR_o, a small number, to R_o. ΔR_o, in turn, is computed by dividing $\Delta P_o = (0.5 - P_o)$ by the first partial

derivative of P with respect to R evaluated at $R = R_o$.* ΔP_o is the error in probability resulting from the first and incorrect estimate for the SEP. See Fig. 4.1.

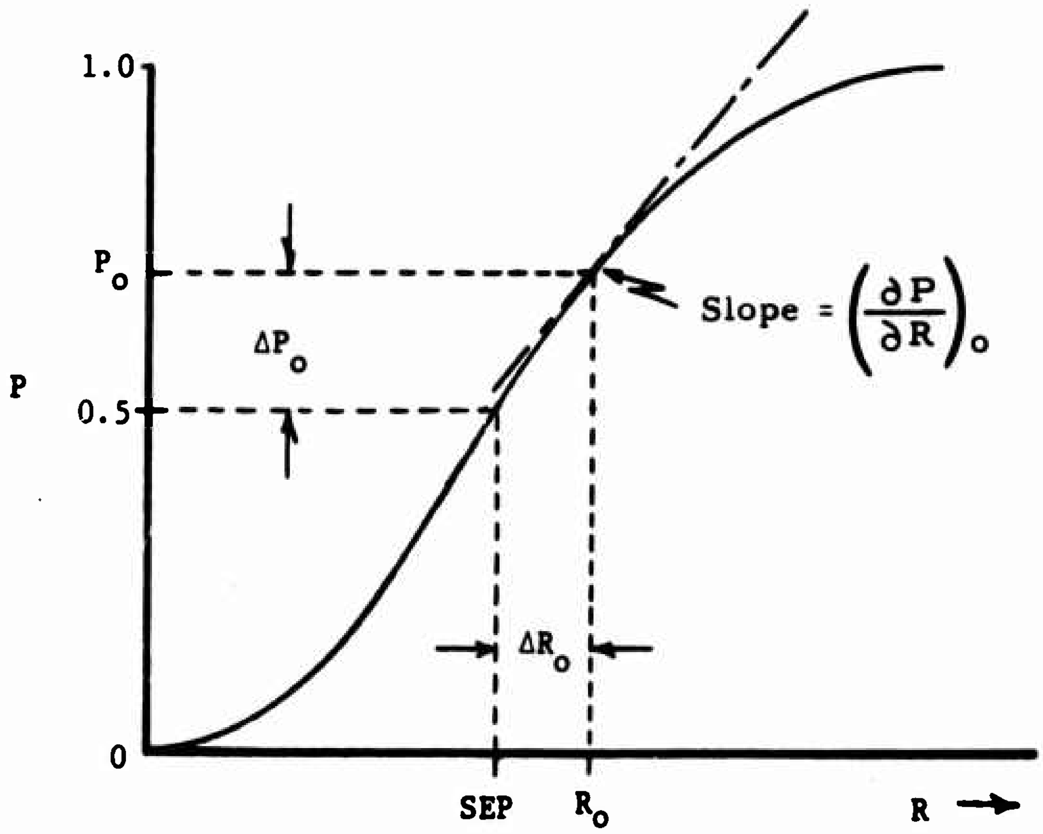

Figure 4.1 A graph of P versus R showing the quantities used in estimating the SEP

The partial derivative, $(\partial P/\partial R)_o$, gives the slope of the probability curve at $R = R_o$. A straight line representation of this slope appears on Fig. 4.1. ΔP_o divided by the $(\partial P/\partial R)_o$ gives the approximate change in R_o required to change P_o by ΔP_o.

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* P_o is the probability calculated for $R = R_o$.

The third estimate for the SEP is made by evaluating the $\partial P/\partial R$, P, ΔP and ΔR at $R = R_1$. R_2 is computed from the expression

$$R_2 = R_1 + \Delta R_1 \tag{4-4}$$

The fourth, fifth and successive estimates are made in an analogous manner until the i^{th} estimate causes the probability to be 0.5. The probability, P, and the partial derivative, $\partial P/\partial R$, are calculated from Lilliefors' series solution of Eq. (1-1). The mathematical details appear in Appendix B.

4.5 Summary of the Computer Method

The computer method produces an exact SEP for any given set of standard deviations. If the accuracy requirement, however, is not too stringent, the SEP can be also found by use of normalized plots. These plots, generated from computer data, display the ratio SEP/σ_z as a function of the ratio σ_x/σ_z. We have produced a set of these plots in Sec. 5. They constitute our fourth and last method for finding the SEP.

SECTION 5

THE FOURTH SOLUTION:
A GRAPHICAL METHOD OF FINDING THE SEP

5.1 The Graph

One SEP can be computed by the method of Sec. 4 for any set of standard deviations, σ_x, σ_y and σ_z. If a large number of SEPs are calculated, they can be tabled as in Fig. 5.1.

σ_x (ft)	σ_y (ft)	σ_z (ft)	SEP (ft)
1	1	1	1.538
1	.5	.55	1.045
1	1	.25	.908
.	.	.	.
.	.	.	.
.	.	.	.

Figure 5.1 A sample tabulation of the SEP as a function of σ_x, σ_y and σ_z.

The tabulation is useful for identifying the SEP associated with a given discrete set of standard deviations. The tabulation, however, reveals no obvious method of interpolating for the SEP associated with a set of values, σ_x, σ_y and σ_z, that is not tabled. Much more information can be derived by plotting the SEP as a function of the standard deviations.

In Fig. 5.2 we have plotted the ratio SEP/ σ_z versus the ratio σ_x/σ_z with σ_y as a parameter.* The curves are seen to be smooth and well-behaved. Given values of σ_x, σ_y and σ_z, these curves can be used in a direct solution for the SEP. No iterative calculations are required. The user need only assign values to the standard deviations, form the ratios σ_x/σ_z and σ_y/σ_x, locate the appropriate curve on the graphs and read off the ratio SEP/σ_z. Since σ_z is known, the SEP can be computed directly from the ratio SEP/σ_z by multiplication; i.e.,

$$\mathrm{SEP} = \left(\frac{\mathrm{SEP}}{\sigma_z}\right) \sigma_z \qquad (5\text{-}1)$$

In the next section we work a sample problem to show how these curves could have been used in the satellite rendezvous problem of Sec. 1.

5.2 A Sample Problem

In Sec. 1 we described an orbiting laboratory travelling in a 100 nm circular orbit. We said that the laboratory was to be resupplied by a second vehicle, the transporter, which would rendezvous and dock with the laboratory. The rendezvous required a

* The curves are patterned on the CEP/σ_1 curves used by R. A. Moore in Ref. 3. The curves were drawn from computer data displayed in Appendix C.

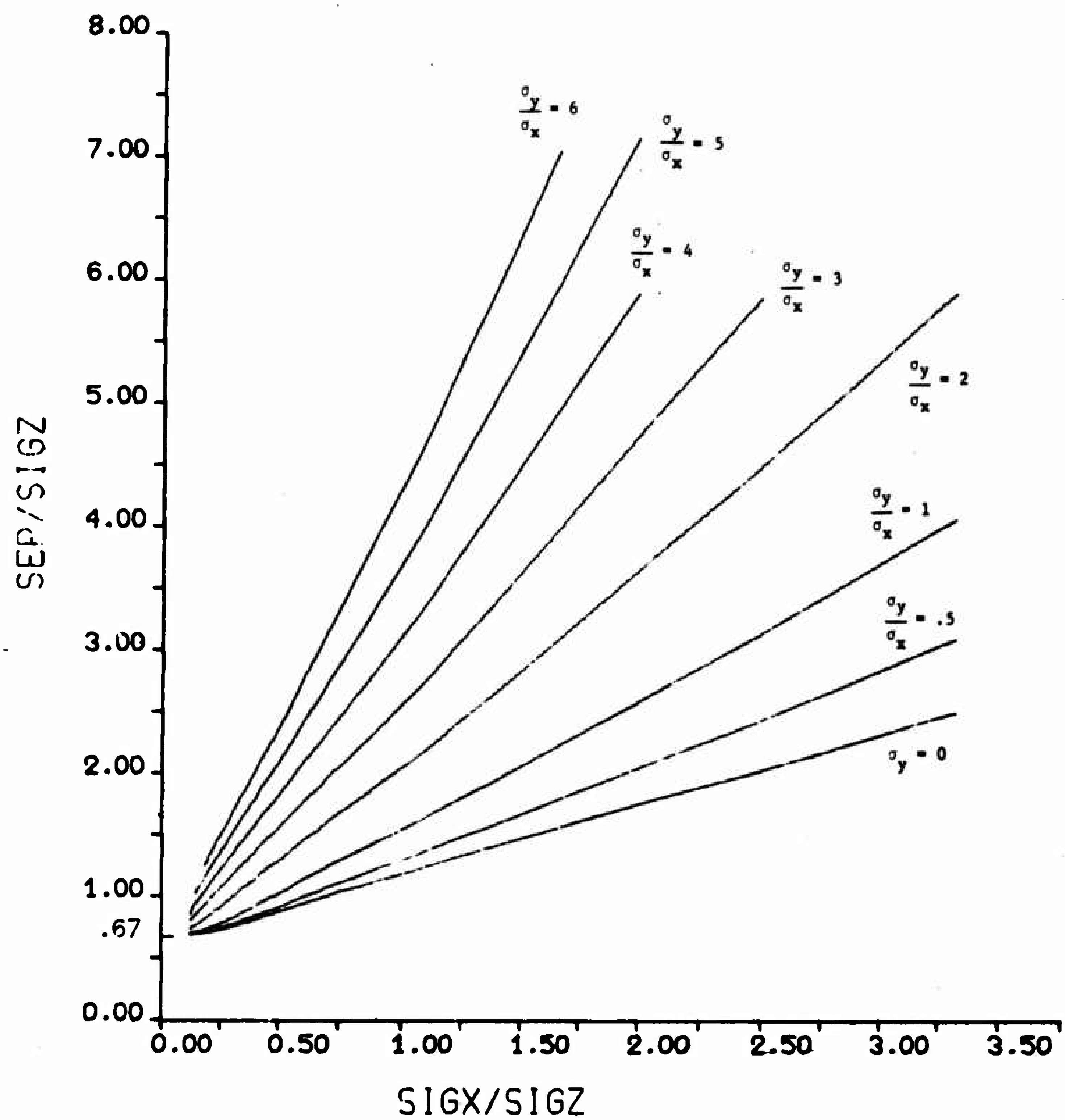

Figure 5.2 A parametric graph for computing the SEP given σ_x, σ_y and σ_z.

precise prediction of the laboratory's position at 2117 Greenwich Mean Time (Zulu). The prediction included a SEP of 8000 feet.

If the mission planners had computed the SEP by using the curves in Fig. 5.2, they would first have used three standard deviations to describe the uncertainties in the laboratory's predicted position. If the deviations were

$$\sigma_x \text{ (Downrange)} = 6350 \text{ ft.} \tag{5-2}$$

$$\sigma_y \text{ (Crossrange)} = 6350 \text{ ft.} \tag{5-3}$$

$$\sigma_z \text{ (Altitude)} = 2540 \text{ ft.} \tag{5-4}$$

at 2117 Zulu, the planners, next, would have formed the ratios σ_x/σ_z and σ_y/σ_x.

$$\sigma_x/\sigma_z = 2.5 \tag{5-6}$$

$$\sigma_y/\sigma_x = 1 \tag{5-7}$$

Entering Fig. 5.2 at $\sigma_x/\sigma_z = 2.5$, they would have proceeded vertically upward to the curve for $\sigma_y/\sigma_x = 1$. On this curve they would have read, for the ratio SEP/σ_z, the value 3.15. Multiplying this value by $\sigma_z = 2540$ ft. would have produced the SEP.

$$\text{SEP} = 3.15 \times 2540 \text{ ft.} = 8000 \text{ ft.} \tag{5-8}$$

The same procedure can be used in computing any SEP. The curves are not restricted to problems in orbital mechanics. A SEP can be calculated in any three-dimensional uncertainty problem where we can give values to the normal variances σ_x^2, σ_y^2 and σ_z^2.

5.3 Application of the Graphical Solution

In the graphical method of finding a SEP, we assign values to the standard deviations σ_x, σ_y and σ_z. These values are generated by our study of the physics and statistics in the actual problem. In general, these values will be associated with a problem coordinate system, like x_1, x_2 and x_3, and not with the coordinates x, y and z. The question arises as to how the values computed for the three variances in x_1, x_2 and x_3 space should be assigned to σ_x^2, σ_y^2 and σ_z^2.

The solution for the SEP is independent of the order of values assigned to σ_x, σ_y and σ_z because Eq. (1-1) is symmetric. The equation is symmetric in the sense that the six combinations of any three values all have the same SEP. See Fig. 5.3 for one example.

σ_x (ft)	σ_y (ft)	σ_z (ft)	SEP (ft)
1	2	3	3.105
1	3	2	3.105
2	3	1	3.105
2	1	3	3.105
3	1	2	3.105
3	2	1	3.105

Figure 5.3 A table showing the symmetry in the SEP calculation.

The user's computed standard deviations, σ_1, σ_2 and σ_3, can be assigned in any order to σ_x, σ_y and σ_z. The same SEP will result for every combination. The graphical solution, however, has widest application when the largest computed deviation is assigned to σ_z. If the largest standard deviation is not assigned to σ_z, but to σ_x instead, the ratio σ_x/σ_z may be larger than 3.5 and beyond the curves of Fig. 5.2.

Finally, we note that the SEP for $m\sigma_x$, $m\sigma_y$ and $m\sigma_z$ is just m times the SEP computed for σ_x, σ_y and σ_z. This result is not immediately obvious from Eq. (1-1) but is apparent in Fig. 5.2. If, for example, the standard deviations, $\sigma_x = 5$ ft, $\sigma_y = 10$ ft, and $\sigma_z = 2.5$ ft, are all doubled, the SEP goes from 3.67 ft. to 7.34 ft.

SECTION 6

SUMMARY

In the opening paragraph of Section 1 we advertised four methods of solving Eq. (1-1) for the Spherical Error Probable. We have described those four methods in Sections 2 through 5. We have showed how a SEP can be calculated by approximation, paper and pencil, computer, or graph.

Our four methods of computing the SEP are not distinctly different. Lilliefors' solution of the probability integral, for example, underlies each of the methods presented in Sections 3, 4 and 5. The same iterative procedure is used in the search for the SEP. We have identified the methods separately, however, because different computing tools - paper and pencil, computer or graph - are used in finding the SEP. The associated computation times also differ markedly.

This paper is not an exhaustive survey of methods for computing the SEP. There probably exist more efficient computer solutions to the probability integral, Eq. (1-1), for example. We have only described, herein, four methods which the authors have found useful in making many repetitive SEP calculations for multiple or time varying sets of standard deviations.

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APPENDIX A

ESTIMATING THE SEP WHEN $\sigma_y = m\sigma_x$

A.1 Outline of the Estimation Method

In Sec. 2 we showed how the SEP (or SEP/ σ_z) could be estimated in the special case where two of the standard deviations were equal. In this appendix we develop a method for estimating the SEP when $\sigma_y = m\sigma_x$, $m \neq 0, 1$. The method uses the SEP/σ_z curve presented in Fig. 2.1.

For purpose of example we will only do calculations for the case when $\sigma_y = 2\sigma_x$. Similar calculations, however, can be applied when $m = 0.5, 3, 4, 5 \ldots$. In the following paragraphs we approximate the SEP/σ_z curve for $\sigma_y = 2\sigma_x$ by calculating the SEP when

(1) σ_x is small compared to σ_z.

(2) σ_x is large compared to σ_z.

(3) σ_x equals σ_z.

(4) σ_x equals $0.5\sigma_z$.

A.2 The SEP for $\sigma_y = 2\sigma_x$ and σ_x Small

In Sec. 2.2 we noted that the three-dimensional probability distribution looked like a one-dimensional distribution when σ_y and σ_x were equal and much smaller than σ_z. The same situation occurs when $\sigma_y = 2\sigma_x$ and σ_x is small. In the limit, as σ_x/σ_z approaches

zero, the probability density function becomes one-dimensional in the variate z. The SEP is given by the equation:

$$\text{SEP} = 0.674\sigma_z \qquad \text{(A-1)}$$

when $\sigma_x/\sigma_z = 0$. In normalized form the ratio, SEP/σ_z, is just 0.674. This is the same value calculated in Sec. 2.2. $\text{SEP}/\sigma_z = 0.674$ is plotted as a point at $\sigma_x/\sigma_z = 0$ on Fig. A.2.

A.3 The SEP for $\sigma_y = 2\sigma_x$ and σ_x Large

For $\sigma_y = 2\sigma_x$ and σ_x large compared to σ_z, the density function, f(x, y, z) looks like a two-dimensional distribution in x and y. The SEP is equal to the CEP defined in Sec. 2.3. The SEP is calculated from Eq. (A.2):

$$\text{SEP} = \text{CEP} = 0.59\,(\sigma_x + \sigma_y) \qquad \text{(A-2)}$$

If we substitute $2\sigma_x$ for σ_y, add, and then divide by σ_z, the familiar normalized function, SEP/σ_z, results:

$$\frac{\text{SEP}}{\sigma_z} = 1.77\,\frac{\sigma_x}{\sigma_z} \qquad \text{(A-3)}$$

Equation (A-3) says that when $\sigma_y = 2\sigma_x$ and σ_x/σ_z is much greater than 1, the SEP/σ_z curve approaches a straight line. The line has a slope of 1.77 and a zero intercept at $\sigma_x/\sigma_z = 0$. This line is plotted on Fig. A.2.

A.4 The SEP for $\sigma_y = 2\sigma_x = 2\sigma_z$

When $\sigma_y = 2\sigma_x$ and σ_x equals σ_z, the SEP can be estimated by using Fig. 2.1 and interchanging the subscripts on the standard deviations. Figure 2.1 was plotted under the assumption that $\sigma_y = \sigma_x$. In this new case we have, instead, σ_z equal to σ_x. If y and z are interchanged wherever they appear on Fig. 2.1, we get Fig. A.1 and a direct solution for the SEP.

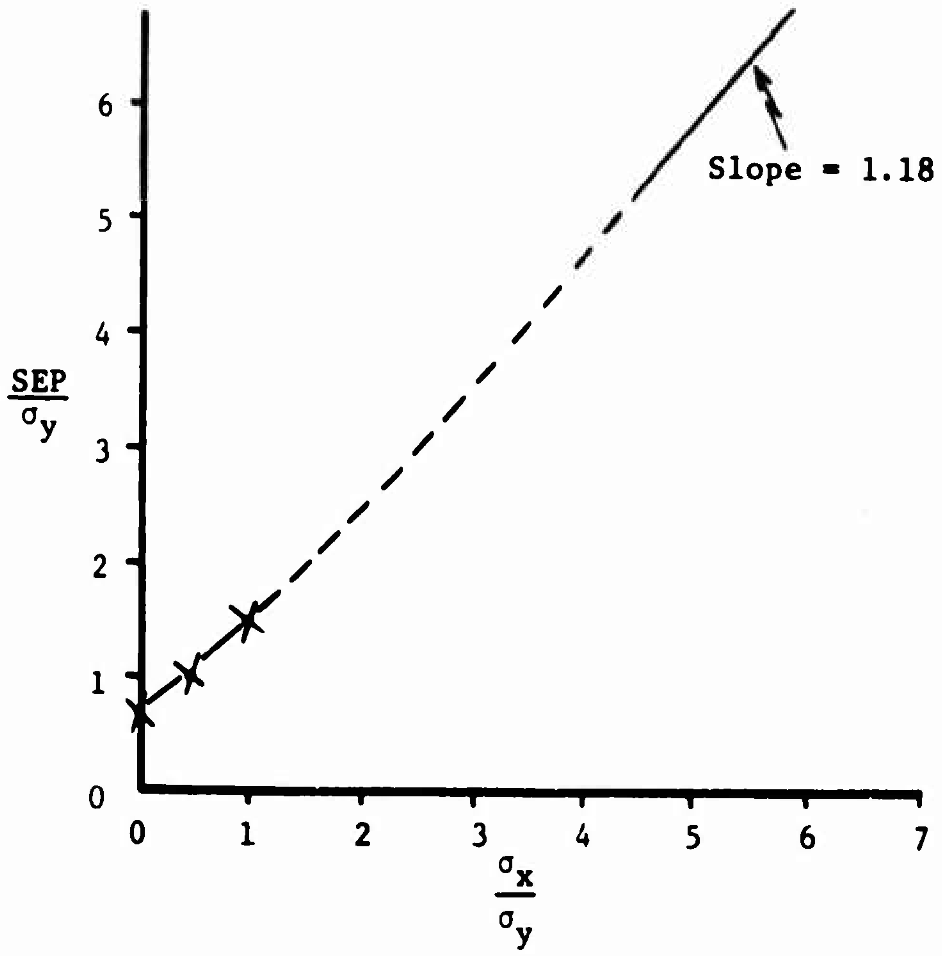

Figure A.1. The predicted SEP/σ_y curve when $\sigma_x = \sigma_z$

At the point where $\sigma_x/\sigma_y = 1/2$ we read that:

$$\text{SEP}/\sigma_y = 1.5382 \qquad \text{(A-4)}$$

Multiplying Eq. (A-4) by the ratio $\sigma_y/\sigma_z = 2$ produces our familiar normalized function, SEP/σ_z, and we have

$$SEP/\sigma_z = 2 \times 1.5382 = 2.0764 \qquad \text{(A-5)}$$

This point is plotted on Fig. A.2 at $\sigma_x/\sigma_z = 1$.

A.5 <u>The SEP when $\sigma_y = \sigma_z = 2\sigma_x$</u>

A final point can be found on the SEP curve for $\sigma_y = 2\sigma_x$ when $\sigma_x = 0.5\sigma_z$. As in Sec. A.4 we find this point by re-plotting Fig. 2.1. The plot is not reproduced here. We interchange the x and z subscripts on the standard deviations and find that the $SEP/\sigma_x = 2.58$ at $\sigma_z/\sigma_x = 2$.

The function, SEP/σ_x, is changed to our standard form through multiplication by the ratio $\sigma_x/\sigma_z = 1/2$. Eq. (A-6) results.

$$\frac{SEP}{\sigma_z} = \left(\frac{\sigma_x}{\sigma_z}\right)\left(\frac{SEP}{\sigma_x}\right) = \left(\frac{1}{2}\right) \cdot \left(2.58\right) = 1.29 \qquad \text{(A-6)}$$

The point, $SEP/\sigma_z = 1.29$, is plotted on Fig. A.2 at $\sigma_x/\sigma_z = 1/2$. We now have sufficient data to draw in the rest of the SEP/σ_z curve.

A.6 <u>The SEP/σ_z Curve for $\sigma_y = 2\sigma_x$</u>

The points and the line found in Sec. A.2 through A.5 appear on Fig. A.2.

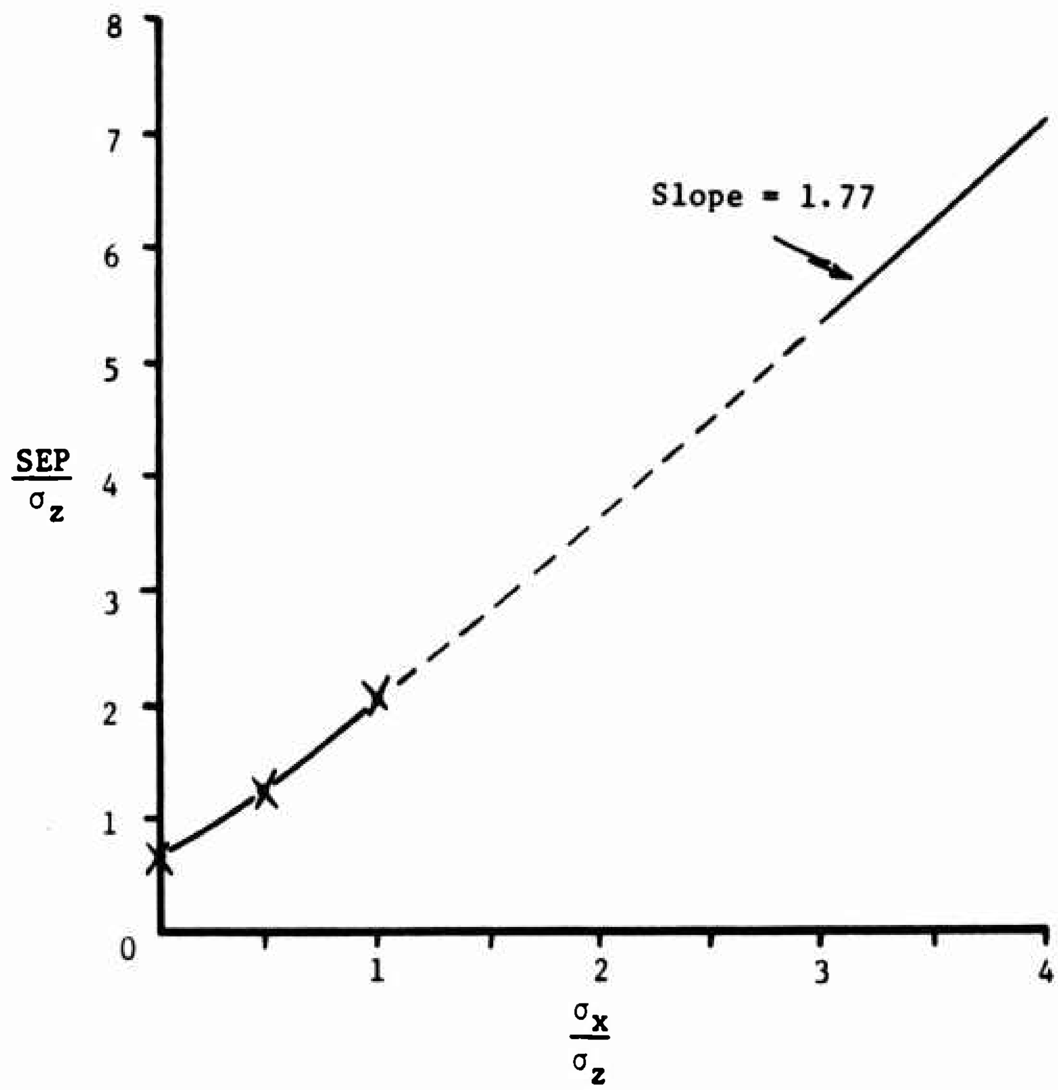

Figure A.2. The predicted SEP/σ_z curve for $\sigma_y = 2\sigma_x$

A dotted line is drawn to connect the points and complete the curve. The SEP can be estimated from this curve in the special case when $\sigma_y = 2\sigma_x$.

By analogous methods the function SEP/σ_z can be graphed for any combination of σ_y and σ_x; i.e., for $\sigma_y = m\sigma_x$; m = 0.5, 2, 3, ... When m = 4, for example, the approximate SEP/σ_z curve is generated by using the curves for m = 0, 1, 2, and 3.

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APPENDIX B

THE MATHEMATICS IN THE ITERATIVE SOLUTION FOR THE SEP

B.1 Lilliefors' Solution for P

Our iterative solution for the SEP is based on a series solution to the integral equation:

$$P = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\int_{-R}^{R}\int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}}\int_{-\sqrt{R^2-z^2-y^2}}^{\sqrt{R^2-z^2-y^2}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}+\frac{z^2}{\sigma_z^2}\right)\right] dx\, dy\, dz \tag{B-1}$$

The series solution was reported by H. W. Lilliefors in Ref. 2. For $R = 1$ and $\sigma_x \neq \sigma_y \neq \sigma_z$, Lilliefors' solution for P has the form:*

$$P = \sqrt{2/\pi}\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}\alpha_1(C_2,n)}{2^{(n+1)}\sigma_y^{(2n-1)}\sigma_x\sigma_z}\sum_{j=0}^{\infty}\left\{\left[\frac{(-1)^j}{j!(2\sigma_z^2)^j}\right]\left[\frac{1}{(j+n+\frac{1}{2})\ldots(j+\frac{1}{2})}\right]\right\} \tag{B-2}$$

* For $\sigma_x = \sigma_y = \sigma_z = \sigma$, the probability integral can be solved by a series expansion of the integrand in Eq. (A-1). For this case, J. S. Toma has reported that the SEP equals 1.5382 .[1] The same answer is derived from Lilliefors' Eq. (B-2).

This solution is correct for $\sigma_z \neq 0$. If $\sigma_z = 0$, P is computed from the expression:

$$P = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} \alpha_1 (C_2, n)}{2^n \sigma_y^{(2n-1)} \sigma_x n!} \qquad \text{(B-3)}$$

The factor, $\alpha_1 (C_2, n)$, appears in both Equations (B-2) and (B-3). $\alpha_1 (C_2, n)$ is calculated from the following recursion formulas:

$$\alpha_1 (C_2, n) = a\ \alpha_2 (C_2, n) + b\beta_2 (C_2, n)$$

$$\alpha_m (C_2, n) = 1 \quad \text{when } m = n$$

$$\alpha_m (C_2, n) = a\alpha_{m+1} (C_2, n) + b\beta_{m+1}(C_2, n) \quad \text{when } m \neq n$$

$$\beta_m (C_2, n) = 0 \quad \text{when } m = n$$

$$\beta_m (C_2, n) = \left[\frac{m-1}{m}\right] \left[b\alpha_{m+1}(C_2, n) + a\beta_{m+1}(C_2, n)\right] \quad \text{when } m \neq n$$

$$\beta_1 (C_2, n) = 0$$

The constants a and b in the recursion formulas are defined by the equations:

$$a = 1 + \frac{1}{2} (k^2 - 1)^{*}$$

$$b = \frac{1}{2} (k^2 - 1)$$

$$k = \frac{\sigma_y}{\sigma_x}$$

---

* This expression for "a" is a correction to the expression in Ref. 2. The equation in Ref. 2 is: $a = 1 + \frac{1}{4\sigma_x^2} (1-z^2)$.

B.2 The Partial Derivative

The iterative solution for the SEP requires successive estimates for R, the spherical radius. The computer program calculates the successive estimates by using the first partial derivative of P with respect to R. In this section we show how the partial derivative is calculated from Lilliefors' series solution for P.

Lilliefors' solution, Eq. (B-2), can be rewritten in the form:

$$P = C_1 \sum_n \frac{Q(n)}{\sigma_y^{(2n-1)} \sigma_x \sigma_z} \cdot \sum_j \frac{N(n, j)}{(\sigma_z^2)^j} \tag{B-4}$$

The standard deviations, σ_x, σ_y and σ_z are normalized in this expression. If σ_x, σ_y and σ_z are redefined to be the non-normalized deviations, Eq. (B-5) results:

$$P = C_1 \sum_n \frac{Q(n)}{\left(\frac{\sigma_y}{R}\right)^{(2n-1)} \left(\frac{\sigma_x}{R}\right)\left(\frac{\sigma_z}{R}\right)} \cdot \sum_j \frac{N(n, j)}{\left(\frac{\sigma_z^2}{R^2}\right)^j} \tag{B-5}$$

We next create Eq. (B-6) by rearrangement of Eq. (B-5).

$$P = C_1 \sum_n \frac{R^{(2n+1)}\, Q(n)}{\sigma_y^{(2n-1)} \sigma_x \sigma_z} \cdot \sum_j \frac{R^{2j}\, N(n, j)}{\sigma_z^{2j}} \tag{B-6}$$

Lilliefors' solution is now in a form where we can take the derivative. Before taking the derivative, however, it is instructive to do

a partial expansion of Eq. (B-6). The first three terms in both series appear in Eq. (B-7).

$$P = \frac{C_1 R^3 Q(1)}{\sigma_y \sigma_x \sigma_z}\left[N(1,0) + N(1,1)\frac{R^2}{\sigma_z^2} + N(1,2)\frac{R^4}{\sigma_z^4} \cdots\right]$$

$$+ \frac{C_1 R^5 Q(2)}{\sigma_y^3 \sigma_x \sigma_z}\left[N(2,0) + N(2,1)\frac{R^2}{\sigma_z^2} + N(2,2)\frac{R^4}{\sigma_z^4} \cdots\right]$$

$$+ \frac{C_1 R^7 Q(3)}{\sigma_y^5 \sigma_x \sigma_z}\left[N(3,0) + N(3,1)\frac{R^2}{\sigma_z^2} + N(3,2)\frac{R^4}{\sigma_z^4} \cdots\right]$$

$$\vdots \qquad \text{(B-7)}$$

If we now take the first partial derivative of P with respect to R, Eq. (B-8) is formed:

$$\frac{\partial P}{\partial R} = \frac{C_1 Q(1)}{\sigma_x \sigma_y \sigma_z} 3R^2\left[N(1,0) + N(1,1)\frac{R^2}{\sigma_z^2} + N(1,2)\frac{R^4}{\sigma_z^4} \cdots\right]$$

$$+ \frac{C_1 Q(2)}{\sigma_y^3 \sigma_x \sigma_z} 5R^4\left[N(2,0) + N(2,1)\frac{R^2}{\sigma_z^2} + N(2,2)\frac{R^4}{\sigma_z^4} \cdots\right]$$

$$+ \frac{C_1 Q(3)}{\sigma_y^5 \sigma_x \sigma_z} 7R^6\left[N(3,0) + N(3,1)\frac{R^2}{\sigma_z^2} + N(3,2)\frac{R^4}{\sigma_z^4} \cdots\right]$$

$$\vdots$$

$$+\frac{C_1 Q(1) R^3}{\sigma_y \sigma_x \sigma_z}\left[2R\frac{N(1,1)}{\sigma_z^{2}} + 4R^3\frac{N(1,2)}{\sigma_z^{2}} \cdots\right]$$

$$+\frac{C_1 Q(2) R^5}{\sigma_y^3 \sigma_x \sigma_z}\left[2R\frac{N(2,1)}{\sigma_z^{2}} + 4R^3\frac{N(2,2)}{\sigma_z^{4}} \cdots\right]$$

$$+\frac{C_1 Q(3) R^7}{\sigma_y^5 \sigma_x \sigma_z}\left[2R\frac{N(3,1)}{\sigma_z^{2}} + 4R^3\frac{N(3,2)}{\sigma_z^{4}} \cdots\right]$$

$$\vdots \qquad \text{(B-8)}$$

Equation (B-8) contains four infinite series. The equation can be rewritten into Eq. (B-9), a compact statement that describes the series in summation form:

$$\frac{\partial P}{\partial R} = C_1 \sum_{n=1}^{\infty} \frac{(2n+1)\, R^{2n}\, Q(n)}{\sigma_y^{(2n-1)} \sigma_x \sigma_z} \bullet \sum_{j=0}^{\infty} \frac{N(n,j)\, R^{2j}}{\sigma_z^{2j}}$$

$$+ C_1 \sum_{n=1}^{\infty} \frac{R^{(2n+1)}\, Q(n)}{\sigma_y^{(2n-1)} \sigma_x \sigma_z} \bullet \sum_{j=0}^{\infty} 2j\, \frac{R^{(2j-1)} N(n,j)}{\sigma_z^{2j}}$$

(B-9)

If we now multiply and divide the first and last series in Eq. (B-9) by R, we produce the expression:

$$\frac{\partial P}{\partial R} = \frac{C_1}{R} \sum_{n=1}^{\infty} \frac{(2n+1)\, R^{(2n+1)} Q(n)}{\sigma_y^{(2n-1)}\, \sigma_x\, \sigma_z} \bullet \sum_{j=0}^{\infty} \frac{N(n,j)\, R^{2j}}{\sigma_z^{2j}}$$

$$+ C_1 \sum_{n=1}^{\infty} \frac{R^{(2n+1)}\, Q(n)}{\sigma_y^{(2n-1)}\, \sigma_x\, \sigma_z} \bullet \frac{1}{R} \sum_{j=0}^{\infty} 2j\, \frac{R^{2j}\, N(n,j)}{\sigma_z^{2j}} \qquad \text{(B-10)}$$

Equation (B-10) can be used to calculate the $\partial P/\partial R$ at any given value of R. For our computer application, however, computation time is saved by noting that Eq. (B-10) resembles Eq. (B-6). In fact, if we rewrite Eq. (B-6) in the form:

$$P = C_1 \sum_n A(n) \bullet \sum_j B(n,j) \qquad \text{(B-11)}$$

we can write Eq. (B-10) as:

$$\frac{\partial P}{\partial R} = C_1 \sum_n \frac{(2n+1)}{R} A(n) \bullet \sum_j B(n,j)$$

$$+ C_1 \sum_n A(n) \bullet \sum_j \frac{2j}{R} B(n,j) \qquad \text{(B-12)}$$

Equations (B-11) and (B-12) show that the two series, $\sum_n A(n)$ and $\sum_j B(n,j)$, are common to the solutions for both P and the $\partial P/\partial R$. Given σ_x, σ_y, σ_z and an estimate for R, both P and $\partial P/\partial R$ can be calculated from only one solution for the two infinite series.

APPENDIX C

A COMPUTER LISTING WITH DATA

C. 1 The SEP Computer Listing

The following CDC 3600 FORTRAN program mechanizes the SEP calculations outlined in Sec. 4.

```
      PROGRAM SEP
C**** THIS PROGRAM COMPUTES THE SEP FOR A PROBABILITY OF .5
      COMMON/S/R,SX,SY,SZ,DR,DX,DY,DZ,FR,FX,FY,FZ,XN,XK,C2
     *,IFLAG
      COMMON/BLK/          SUMP    ,ALPHA(80,80),BETA(80,80),SIGX,SIGY,
     *SIGZ
      COMMON/X/ SIG(300),R5(300),TIM(300),PR(300)
      K=0
      PN=.5
C****READ CARDS
10    CALL READ
       N=XN
      SIZ=SZ
      IF(EOF,60)1000,12
C******SET RADIUS TO BE MIN POSSIBLE VALUE AND YET INSURE CONVERGENCE
12    R=MIN1F(SX/.2,SY/.2,SZ/.2)
      I=0
      IF(SY.EQ.0) GO TO 60
      SIGX=SX/R $ SIGY=SY/R $ SIGZ=SZ/R $ XK=SY/SX $ C2=XK**2-1
      CALL ALPH(N,C2)
13    CALL PROB(N,P,SP)
      IF (IFLAG.NE.0) GO TO 14
      K=K+1
      IF(P.LE..501.AND.P.GE..499.OR.K.GT.20)GO TO 15
      R=R-(P-PN)/SP
      SIGX=SX/R $ SIGY=SY/R $ SIGZ=SZ/R
      GO TO 13
C**** STORE  ONLY THOSE VALUES WHICH CONVERGE
15    IF(K.GT.20) GO TO 14
      I=I+1
C***** STORE VALUES TO BE PUT ON TAPE TO BE PLOTTED
      TIM(I)=I
      PR(I)=P
      R5(I)=R/SZ
      SIG(I)=SX/SZ
155   K=0
      SZ=SZ+DZ
      SIGZ=SZ/R
      IF(SZ.LE.FZ)GO TO 13
```

```
C***  WRITE  TAPE
C**** WRITE ON 49
16    WRITE (10,8),I
      WRITE(10,9)
      WRITE(10),(TIM(K),SIG(K),R5(K),K=1,I)
      END FILE  10
      PRINT 2,SX,SY
      PRINT 1,(SIG(K),R5(K),PR(K),K=1,I)
2     FORMAT (1H1,//30X,*SIGX = *,E20.10, 10X,*SIGY = *,E20.10)
1     FORMAT (10X,*SIGX/SIGZ*,11X,*CEP/SIGZ*,12X,*PROBABILITY*,/,
     *(3F20.8))
      SZ=SIZ
      SY=SY+DY
      IF(SY.LE.FY)GOTO 12
      GO TO 10
14    R=MIN1F (SX/.2,SY/.2,SZ/.2)
      SIGX=SX/R$SIGY=SY/R
      GO TO 155
60    R=MIN1F(SZ/.2,SX/.2,1.)
      KK=0
      SIGX=SX/R
      SIGZ=SZ/R
61    XK=SZ/SX
      C2=XK**2-1
      CALL ALPH(N,C2)
62    FN=1
      SJ1=-1
      PY=PPY=0
      DO 65 K=1,N
      SJ1=-1*SJ1
      FN=FN*K
      SI=SJ1*ALPHA(1,K)/(FN*2.**K*SIGZ**(2*K-1)*SIGX)
      SIP=SI*2 *K/R
      PY=PY+SI
      IF(ABSF(SI/PY).LT.1E-8) GO TO 66
65     FPY=PPY+SIP
66    IF(PY.GT..501.OR.PY.LT..499.AND.KK.LT.20) GO TO 70
      IF(KK.GE.20) GO TO 69
      I=I+1
      R5(I)=R/SZ
      SIG(I)=SX/SZ
      TIM(I)=I
      PR(I)=PY
 69   SZ=SZ+DZ
      IF(SZ.GT.FZ) 16,60
70    R=R-(PY-.5)/PPY
      KK=KK+1
      SIGX=SX/R
      SIGZ=SZ/R
      GO TO 62
1000  RETURN
8     FORMAT (*ID*,52X,*048000003*I3,6X)
9     FORMAT(*RADIUSSIG                                    *)
      END
```

```
      FUNCTION SUMJ(I)
      COMMON/BLK/          SUMP    ,ALPHA(80,80),BETA(80,80),SIGX,SIGY,
     *SIGZ
      COMMON/S/R(15)
     *,IFLAG
      SUMJ=SUMP=0
      SJ1=-1$FJ=1
      DO 20 JJ=1,100
      J=JJ-1
      DJ=J+.5
      DO 10 K=1,I
10    DJ=(J+K+.5)*DJ
      FJ=FJ*J
       IF(FJ.EQ.0.)FJ=1
      SJ1=-1*SJ1
      S    =1/(FJ*(2*SIGZ**2)**J*DJ)*SJ1
      IF(SUMJ  .NE.0.AND.ABSF(S/SUMJ  ).LT.1E-8) GO TO 21
      PS=S*2*J/R
      SUMP   =SUMP    +PS
20    SUMJ  =SUMJ  +S
      IFLAG =1
21    CONTINUE
1     FORMAT (//,(5E24.10))
      RETURN
      END
```

```
      SUBROUTINE PROB(N,P,SP      )
      COMMON/BLK/          SUMP    ,ALPHA(80,80),BETA(80,80),SIGX,SIGY,
     *SIGZ
      COMMON/S/R(15)
     *,IFLAG
      DATA(TWO PI =.7978845608    )
      SP=SN=0
      IFLAG=0
      SJ1=-1
      DO 10 I=1,N
      SJ1=-1.*SJ1
      S=ALPHA(1,I)/(2.**(I+1)*SIGY**(2*I-1)*SIGX*SIGZ)*SJ1
      ST=S*SUMJ(I)
      IF(IFLAG.NE.0) GO TO 9
      IF(SN.NE.0.AND.ABSF(ST/SN).LT.1E-8)GO TO11
       SN=SN+ST
10    SP=SP+SUMP   *S*S*(2*I+1)/R   *ST/S
9     IFLAG=1
11    SP=TWOPI*SP
      P=TWO PI*SN
      RETURN
      END
```

```
      SUBROUTINE ALPH(N,C2)
      COMMON/BLK/           SUMP    ,ALPHA(80,80),BETA(80,80),SIGX,SIGY,
     *SIGZ
      A=C2*.5-1
      B=C2*.5
      M=N
      DO 9  I=1,N
      ALPHA (I,I)=1
9     BETA(I,I)=0
      DO 10  K=1,N
      M=M-1
      DO 10 I=1,M
      ALPHA(I,I+K)=A*ALPHA(I+1,I+K)+B*BETA(I+1,I+K)
10    BETA(I,I+K)=(I-1)/I*(B*ALPHA(I+1,I+K)+A*BETA(I+1,I+K))
      RETURN
      END
```

```
      SUBROUTINE READ
      COMMON/S/R(13),XK,C2
      READ 1,(R(I),I=1,13)
1     FORMAT(5E16.0)
      END
```

C.2 The SEP Computer Data

The following computer data were used in plotting the normalized curves in Fig. 5.2.

SIGX = 1.0000000000+000 SIGY = 0.0000000000+000

SIGX/SIGZ	SEP/SIGZ	PROBABILITY
5.00000000	3.52434371	0.49933981
3.33333333	2.49730070	0.49946628
2.50000000	2.01914995	0.49985782
2.00000000	1.74079638	0.49998594
1.66666667	1.55608473	0.49999961
1.25000000	1.32148778	0.49967359
1.00000000	1.17740951	0.49999970
0.71428571	1.00505711	0.49999860
0.50000000	0.87033511	0.49993984
0.40000000	0.80761956	0.49982800
0.30303030	0.75115376	0.49971807
0.20000000	0.70566641	0.49987442
0.12500000	0.68627760	0.49999826

SIGX = 1.0000000000+000 SIGY = 5.0000000000-001

SIGX/SIGZ	SEP/SIGZ	PROBABILITY
5.00000000	4.47047233	0.49999214
3.33333333	3.08410623	0.49971674
2.50000000	2.42161704	0.49951873
2.00000000	2.03818181	0.49948691
1.42857143	1.60964950	0.49960197
1.00000000	1.28884041	0.49970528
0.50000000	0.90782976	0.49980035
0.30303030	0.76708754	0.49987596
0.21276596	0.70835663	0.49943213

SIGX = 1.0000000000+000 SIGY = 1.0000000000+000

SIGX/SIGZ	SEP/SIGZ	PROBABILITY
3.33333333	4.05597236	0.49999894
2.50000000	3.12140967	0.49988051
2.00000000	2.57805305	0.49982290
1.66666667	2.22537530	0.49980750
1.11111111	1.65177576	0.49985646
0.50000000	1.01956866	0.49986146

0.30303030	0.81937779	0.49987971
0.20408163	0.73849930	0.49993855
0.12500000	0.69841416	0.49995652

SIGX = 1.0000000000+000 SIGY = 2.0000000000+000

SIGX/SIGZ	SEP/SIGZ	PROBABILITY
3.33333333	5.88984063	0.49993903
2.50000000	4.47031812	0.49996898
2.00000000	3.63206608	0.49994283
0.42857143	2.70305127	0.49987570
1.00000000	2.03910945	0.49985054
0.50000000	1.28917711	0.49991805
0.25000000	0.90802420	0.49994871
0.20000000	0.83356676	0.49995688
0.12500000	0.73662162	0.49996220

SIGX = 1.0000000000+000 SIGY = 3.0000000000+000

SIGX/SIGZ	SEP/SIGZ	PROBABILITY
2.50000000	5.84261733	0.49921362
2.00000000	4.72149463	0.49997504
1.66666667	3.97779948	0.49996857
1.00000000	2.54540426	0.49989433
0.50000000	1.55325508	0.49992435
0.30303030	1.16571314	0.49996313
0.20000000	0.95555704	0.49997517
0.14925373	0.85103799	0.49999301
0.12500000	0.80578883	0.49981551

SIGX = 1.0000000000+000 SIGY = 4.0000000000+000

SIGX/SIGZ	SEP/SIGZ	PROBABILITY
2.00000000	5.88878514	0.49977376
1.42857143	4.27103567	0.49998488
1.00000000	3.09053778	0.49995657
0.50000000	1.81601062	0.49993425
0.25000000	1.20446927	0.49997776
0.20000000	1.07690612	0.49999859
0.12500000	0.85402313	0.50078231

SIGX = 1.0000000000+000 SIGY = 5.0000000000+000

SIGX/SIGZ	SEP/SIGZ	PROBABILITY
2.00000000	7.14868648	0.49998480

1.42857143	5.14340312	0.49999067
1.00000000	3.67950015	0.49998552
0.50000000	2.08389447	0.49995018
0.30303030	1.50347918	0.49997427
0.20000000	1.19483273	0.50022996
0.14925373	1.02956778	0.49939611

SIGX = 1.0000000000+000 SIGY = 6.0000000000+000

SIGX/SIGZ	SEP/SIGZ	PROBABILITY
1.66666667	7.04731527	0.50021207
1.25000000	5.31645805	0.49999518
1.00000000	4.29908555	0.49999478
0.50000000	2.36071598	0.49996830
0.29411765	1.63776678	0.49997671
0.18867925	1.25671490	0.50046639

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13. ABSTRACT

When the predicted position of a satellite contains normally distributed errors, the position uncertainty can be described by a Spherical Error Probable or SEP. The SEP is calculated by integrating the three-dimensional normal probability density function over a spherical volume. The SEP is set equal to the radius of that volume which contains the satellite with 50% probability. In this report the authors present four methods for integrating the density function and finding the SEP. The three normal variates in the density function are assumed to be independent and unbiased with known variances.

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